

HIGHER SPIN GENERALIZATION OF THE 6-VERTEX MODEL AND MACDONALD POLYNOMIALS

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ABSTRACT. The partition function of the 6-vertex model with Domain Wall Boundary Conditions (DWBC) was given by Izergin [8, 9], in a determinantal form. It is known that for a special value of the parameters it reduces to a Schur polynomial.

In 2006, Caradoc, Foda and Kitanine [2] computed the partition function of the higher spin generalization of the 6-vertex model. In this article, we prove that, for a special value of the parameters, referred to as the combinatorial point, the partition function is in fact a Macdonald polynomial.

INTRODUCTION

The 6-vertex model with domain wall boundary conditions was introduced by Korepin in 1982 [10]. Later, in his 1987 article [9], Izergin solved the recursion relations satisfied by the partition function, proposed by Korepin and the result is a determinantal formula.

An interesting feature of this model is that it has multiple combinatorial interpretations, being in bijection with several models like Alternating Sign Matrices, Fully Packed Loops, 2d Ice model and others. For example, Kuperberg [12] used this model to compute the number of Alternating Sign Matrices.

The 6-vertex model is an integrable model, meaning that it possesses an R -matrix which satisfies the Yang-Baxter equation. And the fact that we can compute the partition function is just a consequence of the presence of such a matrix.

To each edge on the 6-vertex model, we associate a spin $1/2$. Then it is natural to search for generalizations of these models, where we replace the representation of the underlying algebra or even the algebra itself (like studying the representations of sl_r). The idea is to construct an R -matrix for the model, that satisfies the Yang-Baxter equation. See for example, the article [15] by Pimenta and Martins.

If we restrict ourselves to sl_2 , there is a systematic way of constructing the R -matrix called Fusion [11, 16]. The idea comes from the simple fact that in the representation theory of sl_2 we can build the spin $\ell/2$ representation through the fusion of ℓ spin $1/2$ representations. This was achieved in the work of Caradoc, Foda and Kitanine [2], which serves as the basis of our paper.

The partition function of the 6-vertex model, as defined in Section 2, is a multivariate polynomial symmetric in two separate sets of variables. It also depends on an extra parameter, normally denoted by q .

If we set $q = e^{2\pi i/3}$, the partition function is simply a Schur polynomial [14], which is symmetric in the two sets of variables as a whole. There are several ways of seeing that, see for example Stroganov's article [17].

In this paper we prove that something similar happens in the higher spin cases. If we set $q = e^{2\pi i/2\ell+1}$, then the partition function is a symmetric polynomial in all variables. Moreover, it is a Macdonald polynomial for a staircase Young diagram.

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In order to prove this, we show that for the specific value of q given above, the partition function satisfies a certain version of the wheel condition, which uniquely defines this function. But the Macdonald polynomials also satisfy the same conditions, as it was shown in the article [4] by Feigin, Jimbo, Miwa and Mukhin. This is sufficient to prove our main theorem.

Outline of the paper. The first three sections are introductory: in Section 1 we give a very brief introduction to Macdonald polynomials, based on Macdonald's book [13]; in Section 2 we define the 6-vertex model, and we compute its partition function; Finally, in Section 3 we generalize it for higher spins.

The two following sections constitute the bulk of the article. In Section 4, we point out some properties of the partition function that will be necessary to our main result and that do not depend on a special value of q . Section 5 is about the properties of the partition function when $q^{2\ell+1} = 1$, which is our main result. We prove that it satisfies the wheel condition and therefore it should be a very specific Macdonald polynomial.

In order to clean the flow of the article we leave the heavier proofs, the polynomiality and the wheel condition, to the appendices. We close the article with some concluding remarks.

1. BRIEF INTRODUCTION TO THE MACDONALD POLYNOMIALS

In this section we give a very short introduction to Macdonald polynomials, for the sake of completeness. A more complete introduction can be found in Macdonald's book [13].

1.1. Young diagrams. A sequence of integers $\lambda = \{\lambda_1, \dots, \lambda_N\}$, in decreasing order, is said to be a partition of n if $\sum_i \lambda_i = n$. It is normally denoted by $\lambda \vdash n$. The elements λ_i are called the parts of λ .

A partition can be represented by a Young diagram. For example, if $\lambda = \{3, 1, 1, 0\}$, then the corresponding Young diagram is:

$$Y(\{4, 1, 1, 0\}) = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array}$$

The size of the Young diagram $Y(\lambda)$ is the number of boxes in the Young diagram and is denoted by $|Y(\lambda)| = \sum_i \lambda_i$.

In what follows we won't distinguish partitions and Young diagrams.

Definition 1.1 (Partial order). We say that $\lambda \leq \mu$ if

$$\sum_{i=1}^j \lambda_i \leq \sum_{i=1}^j \mu_i \quad \text{for all } j.$$

1.2. Bases of symmetric polynomials. Symmetric polynomials form a vector space, and several bases and transformations between them had been extensively studied. These bases are normally indexed by Young diagrams or, equivalently, partitions.

Two of them are relevant to our work.

Definition 1.2 (Monomial basis). Let $\mathbf{x} = \{x_1, \dots, x_N\}$ be a set of variables, eventually infinite¹. Let $\lambda = \{\lambda_1, \dots, \lambda_M\}$ be a partition, where $M \leq N$. Then

$$m_\lambda := \sum'_{\sigma \in S_N} x_{\sigma_1}^{\lambda_1} x_{\sigma_2}^{\lambda_2} \dots x_{\sigma_M}^{\lambda_M}$$

¹In that case, it is a symmetric function.

where the sum is restricted to the elements of the symmetric group S_N that act non-trivially in the monomial $x_1^{\lambda_1} \dots x_M^{\lambda_M}$.

Definition 1.3 (Power sum basis). Let $p_i = \sum_j x_j^i$ and let $\lambda = \{\lambda_1, \dots, \lambda_M\}$ be a partition, where $M \leq N$. Then

$$p_\lambda := \prod_{i=1}^M p_{\lambda_i}$$

The degree of p_λ and m_λ is exactly $|\lambda| = \sum_i \lambda_i$.

1.3. Macdonald polynomials. Let $z_\lambda := \prod_i i^{m_i} m_i!$, where m_i is the number of parts equal to i . Define a scalar product by

$$\langle p_\lambda, p_\mu \rangle_{p,t} = \delta_{\lambda,\mu} z_\lambda \prod_i \frac{1 - p^{\lambda_i}}{1 - t^{\lambda_i}}$$

Notice that the scalar product is positive definite if p and t are real numbers between 0 and 1.

A Macdonald polynomial is a symmetric polynomial in \mathbf{x} , which depends on two extra parameters p and t (in the literature q is used instead of p , but we avoid it because q is also the usual parameter in the 6-vertex model), defined through the following two properties:

$$(1) \quad P_\lambda(\mathbf{x}; p, t) = m_\lambda(\mathbf{x}) + \sum_{\mu < \lambda} c_{\lambda,\mu}(p, t) m_\mu(\mathbf{x})$$

and:

$$(2) \quad \langle P_\lambda(\mathbf{x}; p, t), m_\mu(\mathbf{x}) \rangle_{p,t} = 0 \quad \text{if } \mu < \lambda.$$

this implies that the Macdonald polynomials are orthogonal with respect to the scalar product above:

$$(3) \quad \langle P_\lambda(\mathbf{x}; p, t), P_\mu(\mathbf{x}; p, t) \rangle_{p,t} = 0 \quad \text{if } \mu \neq \lambda.$$

This is enough to define the Macdonald polynomials.

We can obtain other bases by specializing the Macdonald polynomials: for example, if we set $p = 0$ we obtain the Hall–Littlewood polynomials; for $p = t^\alpha$ and by letting $t \rightarrow 1$ we obtain the Jack polynomials.

We will need the following proposition:

Proposition 1.4. *The coefficients $c_{\lambda,\mu}(p, t)$ are rational functions on p and t .*

Proof. Let \mathbb{K} be the field of rational functions on two variables p and t . Then $\langle m_\lambda, m_\mu \rangle \in \mathbb{K}$. In order to generate the Macdonald polynomials we apply the Gram–Schmidt process. The result follows. \square

1.4. Schur polynomials. The basis consisting of Schur polynomials $S_\lambda(\mathbf{x})$ can be defined starting from the Macdonald polynomials by setting: $p = t$, i.e.

$$(4) \quad \langle S_\lambda(\mathbf{x}), S_\mu(\mathbf{x}) \rangle = \delta_{\lambda\mu}$$

where the scalar product is defined by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda.$$

There is a more direct way of defining them: let $\lambda = \{\lambda_1, \dots, \lambda_m\}$ and $\mathbf{x} = \{x_1, \dots, x_n\}$, where $n \geq m$. We can suppose that $n = m$, otherwise we add zeros at

the end of λ . Then the Schur polynomial corresponding to the partition λ is given by

$$S_\lambda(\mathbf{x}) = \frac{\det \left| x_i^{m-j+\lambda_j} \right|_{i,j}^m}{\det \left| x_i^{m-j} \right|_{i,j}^m}.$$

2. REVIEW OF THE 6-VERTEX MODEL

In this section we give a brief description of the 6-vertex model. We are interested in a very specific kind of boundary conditions, called Domain Wall Boundary Conditions (DWBC).

It is known that, with these boundary conditions, the 6-vertex model is in bijection with several other models, like the Alternating Sign Matrices or the Fully Packed Loops. Although these connections are interesting, it is out of the scope of this article to describe them in detail.

The description of the model will follow the construction presented in [6].

2.1. Definition of the model. Take a square grid² of size $n \times n$, in which each edge is given an orientation (an arrow), such that at each vertex there are as many arrows pointing in as arrows pointing out. This gives 6 possibilities.

At the boundaries we impose that the arrows at the top and the bottom are pointing outwards and the ones at the left and at the right are pointing inwards. See, for an example, figure 1.

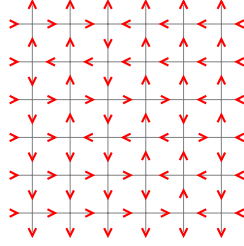


FIGURE 1. A configuration of the 6-vertex model.

To each vertex configuration we give a certain weight depending on a global parameter q and two parameters, called spectral parameters, see figure 2. We need then to introduce n horizontal spectral parameters, $\mathbf{x} = \{x_1, \dots, x_n\}$, one for each row, and n vertical spectral parameters $\mathbf{y} = \{y_1, \dots, y_n\}$, one for each column.

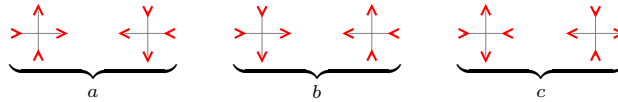


FIGURE 2. To each site configuration a statistical weight described below is associated.

The weights are given by the following formula:

$$\begin{aligned} a(x, y) &= qx - q^{-1}y \\ b(x, y) &= x - y \\ c(x, y) &= (q - q^{-1})\sqrt{xy} \end{aligned}$$

²We do not intend to give the most general definition, just the one that we use.

2.2. The partition function. The weight of a configuration is then defined by the product of the weight of the vertices. The partition function is defined as the sum over all possible configurations:

$$(5) \quad Z_n(\mathbf{x}, \mathbf{y}) := \sum_{\text{configurations}} \prod_{i,j}^n w_{ij}(x_i, y_j),$$

where $w_{ij}(x_i, y_j)$ is replaced by $a(x_i, y_j)$, $b(x_i, y_j)$ or $c(x_i, y_j)$ depending on the vertex configuration.

We renormalize the partition function:

$$\mathcal{Z}_n(\mathbf{x}, \mathbf{y}) = (-1)^{\binom{n}{2}} (q^{-1} - q)^{-n} \prod_{i=1}^n x_i^{-1/2} y_i^{-1/2} Z_n(\mathbf{x}, \mathbf{y})$$

Like this, we obtain a homogeneous polynomial with total degree $\delta = n(n-1)$ (in all spectral parameters) and a partial degree $\delta_i = n-1$ (for each variable x_i or y_i individually).

The partition function was explicitly computed by Izergin [9], using the recursion relations written by Korepin [10]. The result is an $n \times n$ determinant, symmetric in \mathbf{x} and \mathbf{y} . Much later it was observed by Stroganov [18] and Okada [14] that when $q = e^{2i\pi/3}$, the partition function is totally symmetric. Moreover, the partition function is a specific Schur polynomial.

2.3. Integrability. The interest of this model lies in the fact that it is a quantum integrable model meaning that we can construct an R -matrix that satisfies the Yang–Baxter equation (YBE).

Take the $n \times n$ grid, fix the boundaries and imagine at each site an operator that takes two arrows at the left and bottom edges and gives as a result the two arrows at the right and top edges. We can represent it as a matrix:

$$\begin{array}{c} \text{x} \\ \uparrow \\ \boxed{\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array}} \\ \text{y} \end{array} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} =: R(y, x),$$

where the entries follow the order: $\{\rightarrow\uparrow, \rightarrow\downarrow, \leftarrow\uparrow, \leftarrow\downarrow\}$. The drawing at the right is just a graphical way of representing the R -matrix, where the arrows point to the direction of the operation.

We can then think of the arrows as spins, i.e. an arrow on the same direction as the operation is positive and an arrow against it is negative. In other words, an arrow pointing up or right is the same as a spin up (or $1/2$) and an arrow pointing down or left is the same as a spin down (or $-1/2$).

The R -matrix satisfies the following identity equation:

$$(6) \quad R(x, y)R(y, x) = (qy - q^{-1}x)(qx - q^{-1}y)Id.$$

Graphically:

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} = (qy - q^{-1}x)(qx - q^{-1}y) \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array}$$

The main property of the R -matrices is that they satisfy the YBE:

$$R_{2,3}(y_2, y_3)R_{1,3}(y_1, y_3)R_{1,2}(y_1, y_2) = R_{1,2}(y_1, y_2)R_{1,3}(y_1, y_3)R_{2,3}(y_2, y_3),$$

where $R_{i,j}$ means that R acts on the tensor product between the i^{th} and the j^{th} vector space³. Graphically:

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array}$$

³We consider that each line transports a vector space associated to one spin.

We define the transfer matrix of the model by

$$(7) \quad T(x, \mathbf{y}) = {}_0 \langle \uparrow | R_{1,0}(y_1, x) R_{2,0}(y_2, x) \dots R_{n,0}(y_n, 0) | \downarrow \rangle_0$$

where the matrix $R_{i,0}$ acts on the tensor product of the i^{th} space and the so-called auxiliary space. The $|\downarrow\rangle_0$, means that we select the spin down, at the end, and similarly we chose the spin down at the beginning.

Then the partition function is given by:

$$(8) \quad Z_n(\mathbf{x}, \mathbf{y}) = \langle \downarrow \downarrow \dots \downarrow | T(x_1, \mathbf{y}) T(x_2, \mathbf{y}) \dots T(x_n, \mathbf{y}) | \uparrow \uparrow \dots \uparrow \rangle$$

where we impose at the bottom only negative spins and at the top all positive spins.

Using the YBE, we can prove that $[T(x, \mathbf{y}), T(x', \mathbf{y})] = 0$, and therefore $Z_n(\mathbf{x}, \mathbf{y})$ is symmetric in \mathbf{x} . Of course, instead of doing it row by row, we could do it column by column arriving to a similar conclusion.

Without entering into more details, it can be proved that the partition function satisfies some recursion relation and this plus the symmetries and the degree are enough to prove that it is given by a determinant, the so-called Izergin–Korepin determinant:

$$(9) \quad Z_n(\mathbf{x}, \mathbf{y}) = \frac{\prod_{i,j} (x_i - qy_j)(x_i - q^{-1}y_j)}{\prod_{i < j} (x_i - x_j)(y_i - y_j)} \det \left| \frac{1}{(x_i - qy_j)(x_i - q^{-1}y_j)} \right|_{i,j=1}^n$$

where, for the sake of simplicity, we had replaced $y_i \rightarrow qy_i$, making the partition function symmetric under the exchange $\mathbf{x} \leftrightarrow \mathbf{y}$.

The proof is very simple, we only need to prove that this is a polynomial, symmetric in \mathbf{x} and \mathbf{y} that satisfies the Korepin recursion relation. Moreover, we prove that if we have two polynomials satisfying these conditions they must be the same up to a multiplicative constant.

2.4. A Schur polynomial. Let Y_n be the Young diagram $Y_n = (n-1, n-1, n-2, n-2, \dots, 1, 1, 0, 0)$. When $q^3 = 1$, a miracle happens and this big expression becomes a Schur polynomial:

$$Z_n(\mathbf{x}, \mathbf{y}) = s_{Y_n}(\mathbf{x}, \mathbf{y})$$

the proof uses the same method: we only need to prove that it is a homogeneous polynomial with the right degree that satisfies the same recursion relation.

Notice that being a Schur polynomial, the partition function Z_n is fully symmetric.

Because of the fact that we get a symmetric polynomial and because replacing $x_i \rightarrow 1$ and $y_i \rightarrow 1$ we get the enumeration of Alternating Sign Matrices we call the special value $q^3 = 1$ the combinatorial point.

3. FUSION AND THE PARTITION FUNCTION

In this section we explain the fusion process introduced in [11].

3.1. Fusion - Irreducible representations of sl_2 . Let $V^{(\ell)}$ be the $(\ell+1)$ -dimensional irreducible representation of sl_2 (for $\ell = 1$ we omit the superscript). It is a fairly known fact [7] that $V^{(\ell)}$ is the largest irreducible component of $V^{\otimes \ell}$. Therefore we can construct it by taking the tensor product of ℓ representations V and projecting $V^{\otimes \ell} \rightarrow V^{(\ell)}$, this process is known as fusion. In what follows, we repeat the basics just to fix notation and to write down some trivialities.

As usual, we diagonalize S^z , then V is spanned by the two eigenvectors, let them be $|+\rangle$ and $|-\rangle$. This will generate a canonical basis for $V^{\otimes \ell}$. We simplify the notation, for example $|+\rangle \otimes |+\rangle \otimes |-\rangle$ becomes $|++-\rangle$.

Then the space $V^{(\ell)}$ is obtained from the highest state, all pluses, by applying S^- several times. Write

$$|\ell; \ell\rangle := \underbrace{|++\dots+}_{\ell} \quad |\ell; \ell - m\rangle := \frac{1}{m!} (S^-)^m |\ell; \ell\rangle$$

where m goes from zero to ℓ .

Proposition 3.1. *Then the state $|\ell; \ell - m\rangle$ is a sum over all possible configurations with $\ell - m$ pluses and m minuses with coefficient 1.*

For example, for $\ell = 4$ and $m = 2$, we have:

$$|4; 2\rangle = |+++-\rangle + |+-+-\rangle + |+--+\rangle + |-++-\rangle + |-+-+\rangle + |--++\rangle$$

And, as we can easily check:

Proposition 3.2. *A state in $V^{\otimes \ell}$ is in $V^{(\ell)}$ if and only if it is symmetric, i.e. if we exchange position i with $i + 1$ the state remains the same for $1 \leq i \leq \ell - 1$.*

3.2. A higher dimensional R -matrix. In the preceding case we had constructed a R -matrix of the form $R(x, y) : V_1 \otimes V_2 \rightarrow V_1 \otimes V_2$. Now we are interested in the analogous setup for a higher level, i.e. we want to construct

$$R^{(\ell)}(x, y) : V_1^{(\ell)} \otimes V_2^{(\ell)} \rightarrow V_1^{(\ell)} \otimes V_2^{(\ell)}$$

which satisfies the identity equation and the Yang–Baxter equation.

For that we use the fusion process, that is, we construct an operator on $V_1^{\otimes \ell} \otimes V_2^{\otimes \ell}$. Write

$$V^{\otimes \ell} \otimes V^{\otimes \ell} = V_1 \otimes \dots \otimes V_\ell \otimes V_{\ell+1} \otimes \dots \otimes V_{2\ell}$$

To each vector space we associate a spectral parameter:

$$\{x, q^2x, \dots, q^{2\ell-2}x, y, q^2y, \dots, q^{2\ell-2}y\}$$

Definition 3.3. Up to a multiplicative constant, $R^{(\ell)}(x, y)$ is given by:

$$\begin{aligned} R^{(\ell)}(x, y) := & C(x, y) R_{1,2\ell}(x, q^{2\ell-2}y) R_{1,2\ell-1}(x, q^{2\ell-4}y) \dots R_{1,\ell+1}(x, y) \\ & \times R_{2,2\ell}(q^2x, q^{2\ell-2}y) R_{2,2\ell-1}(q^2x, q^{2\ell-4}y) \dots R_{2,\ell+1}(q^2x, y) \\ & \dots \\ & \times R_{\ell,2\ell}(q^{2\ell-2}x, q^{2\ell-2}y) R_{\ell,2\ell-1}(q^{2\ell-2}x, q^{2\ell-4}y) \dots R_{\ell,\ell+1}(q^{2\ell-2}x, y) \end{aligned}$$

where $C(x, y)$ is an arbitrary constant, which we will ignore.

Graphically, this can be represented by:

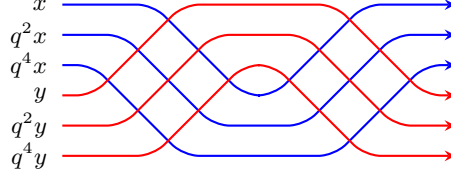
$$R^{(\ell)}(x, y) := \begin{array}{c} \text{Diagram: A box with a vertical red arrow pointing up and a horizontal blue arrow pointing right. The left side is labeled } x \text{ and the bottom is labeled } y. \end{array} = \begin{array}{c} \text{Diagram: A grid of 3 horizontal blue lines and 3 vertical red lines. The horizontal lines are labeled } x, q^2x, q^4x \text{ from top to bottom. The vertical lines are labeled } y, q^2y, q^4y \text{ from left to right.} \end{array}$$

It is now easy to prove that this operator has the properties that we need.

Proposition 3.4. *The operator $R^{(\ell)}(x, y)$ satisfies the identity equation, i.e.*

$$R^{(\ell)}(y, x) R^{(\ell)}(x, y) \propto Id.$$

Proof. In what follows we ignore all multiplicative constants. A graphical proof is easier, let us represent $R^{(3)}(y, x) R^{(3)}(x, y)$, for example:

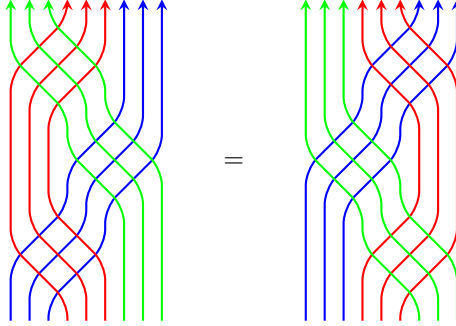


This is clearly the identity. And can be reproduced for any ℓ . \square

Proposition 3.5. *The $R^{(\ell)}$ matrix satisfies the Yang–Baxter equation.*

$$R_{2,3}^{(\ell)}(y_2, y_3)R_{1,3}^{(\ell)}(y_1, y_3)R_{1,2}^{(\ell)}(y_1, y_2) = R_{1,2}^{(\ell)}(y_1, y_2)R_{1,3}^{(\ell)}(y_1, y_3)R_{2,3}^{(\ell)}(y_2, y_3) .$$

Proof. We follow the same procedure as before. Let $\ell = 3$, and represent graphically the Yang–Baxter equation.



The proof is now clear, we only need to apply the Yang–Baxter equation for the small R -matrix several times. \square

Up to now, there is nothing special in the choice of spectral parameters. In fact, these spectral parameters are important because of the following property:

Proposition 3.6. *A state $|v\rangle \in V^{\otimes \ell}$ belongs to $V^{(\ell)}$ if and only if*

$$R_{i,i+1}(q^{2i-2}x, q^{2i}x)|v\rangle = 0$$

for all $1 \leq i \leq \ell - 1$.

Proof. We only need to prove that the operator $R_{i,i+1}(q^{2i-2}x, q^{2i}x)$ annihilate $|v\rangle$ if and only if $|v\rangle$ is invariant under exchanging positions i and $i + 1$. It is a local operator, so we only need to focus on the two concerning positions. We compute the R -matrix:

$$R_{i,i+1}(q^{2i-2}x, q^{2i}x) = (q^2 - 1)q^{2i-2}x \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore the R -matrix annihilates $|v\rangle$ if and only if $|v\rangle$ is stable on the exchange of positions $i \leftrightarrow i + 1$.

By Proposition 3.2, this is equivalent to say that $|v\rangle$ is in $V^{(\ell)}$ if and only if $R_{i,i+1}(q^{2i-2}x, q^{2i}x)$ annihilates $|v\rangle$ for all i . \square

Notice that

$$R_{i,i+1}(q^{2i-2}x, q^{2i}x)(|+-\rangle - |-+\rangle) = C_i(x)(|+-\rangle - |-+\rangle)$$

where $C_i(x)$ is some nonzero scalar function.

We are now ready to state the main result of this section:

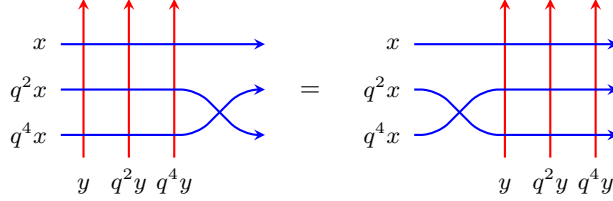
Lemma 3.7. *The operator $R^{(\ell)}$ leaves $V^{(\ell)} \otimes V^{(\ell)}$ invariant and therefore the restricted map*

$$R^{(\ell)}(x, y) : V^{(\ell)} \otimes V^{(\ell)} \rightarrow V^{(\ell)} \otimes V^{(\ell)}$$

is well-defined.

Proof. This is now easy to prove. Let $|v\rangle \otimes |w\rangle \in V^{(\ell)} \otimes V^{(\ell)}$, then we want to prove that $R^{(\ell)}(x, y)|v\rangle \otimes |w\rangle \in V^{(\ell)} \otimes V^{(\ell)}$.

Notice first that we have the following commutation relation:



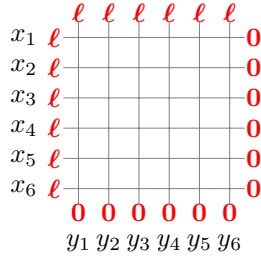
Apply a small R -matrix, suppose $1 \leq i < \ell$, then we have:

$$\begin{aligned} R_{i,i+1}(q^{2i-2}x, q^{2i}x) \left(R^{(\ell)}(x, y)|v\rangle \otimes |w\rangle \right) \\ = R^{(\ell)}(x, y) \left(R_{i,i+1}(q^{2i-2}x, q^{2i}x)|v\rangle \right) \otimes |w\rangle \end{aligned}$$

which vanishes if $|v\rangle$ is in $V^{(\ell)}$. The proof is identical if $\ell < i < 2\ell$, just that it will apply to $|w\rangle$.

Do it for all $1 \leq i < 2\ell$ (exclude $i = \ell$, as well) and by Proposition 3.2 we get the desired result. \square

3.3. Higher spin model. Take a $n \times n$ grid, as in the 6-vertex model, and to each edge we associate a integer number α from 0 to ℓ , which is naturally linked to the states $|\ell; m\rangle$. We fill the left and top boundaries with ℓ , and the bottom and right boundaries with zero. That is:



As before, we add $2n$ spectral parameters $\{\mathbf{x}, \mathbf{y}\}$, as shown in the image.

As in the 6-vertex model, we impose the following conservation condition on a local configuration:

$$\begin{array}{c} \eta \\ \alpha \text{---} \gamma \\ | \\ \beta \end{array} \quad \text{with } \alpha + \beta = \gamma + \eta$$

Naturally, we use the notation:

$$R^{(\ell)}(x_i, y_j)|\ell; \alpha\rangle \otimes |\ell; \beta\rangle = R_{\alpha, \beta}^{(\ell) \gamma, \eta}(x_i, y_j)|\ell; \gamma\rangle \otimes |\ell; \eta\rangle$$

Then, the weights of a vertex configuration are given by:

$$w_{i,j}(x_i, y_j) = R_{\alpha, \beta}^{(\ell) \gamma, \eta}(x_i, y_j)$$

This choice guarantees the integrability of the model.

3.4. Partition function. The partition function is defined as above:

$$(10) \quad Z_{n,\ell}(\mathbf{x}, \mathbf{y}) := \sum_{\text{configurations}} \prod_{i,j}^n w_{i,j}(x_i, y_j)$$

Using the fusion process we obtain an exact expression for the partition function:

Let $\bar{\mathbf{x}} = \{x_1, q^2x_1, \dots, q^{2\ell-2}x_1, \dots, x_n, q^2x_n, \dots, q^{2\ell-2}x_n\}$, and identically for $\bar{\mathbf{y}}$. Then, it is obvious that we get:

$$(11) \quad Z_{n,\ell}(\mathbf{x}, \mathbf{y}) = Z_{\ell n}(\bar{\mathbf{x}}, \bar{\mathbf{y}})$$

The partition function $Z_{n,\ell}(\mathbf{x}, \mathbf{y})$ factorizes; in order to simplify the computations, we remove all these factors:

$$\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y}) = C(q) \frac{\prod_i^n (x_i y_i)^{-\ell/2}}{\prod_{i,j}^n \prod_{p=0}^{\ell-2} \prod_{k=0}^{\ell-1} (q^{2k} x_i - q^{2p+1} y_j)} Z_{n,\ell}(\mathbf{x}, \mathbf{y})$$

where $C(q)$ is some irrelevant function of q , fixed by the definition (12) below.

Using the Vandermonde polynomials in the extended set of variables $\bar{\mathbf{x}}$,

$$\begin{aligned} \Delta(\bar{\mathbf{x}}) &= \prod_{i < j}^{\ell n} (\bar{x}_j - \bar{x}_i) = \prod_i^n \prod_{k < p} (q^{2p} x_i - q^{2k} x_i) \prod_{i < j}^n \prod_{k,p=0}^{\ell-1} (q^{2p} x_j - q^{2k} x_i) \\ &= \prod_{k < p} (q^{2p} - q^{2k})^n \prod_i^n x_i^{\binom{\ell}{2}} \prod_{i < j}^n \prod_{k,p=0}^{\ell-1} (q^{2p} x_j - q^{2k} x_i), \end{aligned}$$

we get the expression

$$(12) \quad \mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y}) := \frac{\prod_{i,j=1}^n \prod_{p=0}^{\ell-1} \prod_{k=0}^{\ell-1} (q^{2k} x_i - q^{2p+1} y_j)}{\Delta(\bar{\mathbf{x}}) \Delta(\bar{\mathbf{y}})} \det \mathcal{A}_\ell(\mathbf{x}, \mathbf{y})$$

where $\mathcal{A}_\ell(\mathbf{x}, \mathbf{y})$ is the $\ell n \times \ell n$ matrix given by:

$$(13) \quad \mathcal{A}_\ell(\mathbf{x}, \mathbf{y}) := [A_\ell(x_\alpha, y_\beta)]_{\alpha,\beta=1}^n$$

and the blocks $A_\ell(x, y)$ are $\ell \times \ell$ matrices of the form

$$(14) \quad A_\ell(x, y) := \left[\frac{1}{(q^{2j}x - q^{2i-1}y)(q^{2j}x - q^{2i+1}y)} \right]_{i,j=0}^{\ell-1}$$

Notice that, if $\ell = 1$, we recover the Izergin–Korepin determinant.

Sometimes it will be useful to use the fact that

$$\mathcal{A}_\ell(\mathbf{x}, \mathbf{y}) = \mathcal{A}_1(\bar{\mathbf{x}}, \bar{\mathbf{y}}).$$

4. SOME PROPERTIES OF THE PARTITION FUNCTION

In this section we state and prove some easy properties of the partition function for the higher spin generalization.

4.1. Polynomiality. We start with a simple statement:

Lemma 4.1. *The partition function $\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y})$ is a homogeneous polynomial on the set of variables $\{\mathbf{x}, \mathbf{y}\}$.*

Proof. We want to compute the determinant of the matrix $\mathcal{A}_1(\bar{\mathbf{x}}, \bar{\mathbf{y}})$. It is a determinant, therefore it is skew symmetric in the variables $\bar{\mathbf{x}}$ and also $\bar{\mathbf{y}}$, we can think of $\bar{\mathbf{x}}$ as a set of ℓn independent variables, and equally for $\bar{\mathbf{y}}$. We can now divide by the two Vandermonde determinants $\Delta(\bar{\mathbf{x}})$ and $\Delta(\bar{\mathbf{y}})$ that explains the denominator of the pre-factor.

Next we want to prove that the poles from the determinant are cancelled by the numerator of the pre-factor. Look at the column of $\mathcal{A}_1(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ corresponding to $q^{2k}x_i$. For each y_j , we can have poles like

$$\frac{1}{(q^{2k}x_i - q^{2p-1}y_j)(q^{2k}x_i - q^{2p+1}y_j)}$$

But the determinant can be written as a sum over the symmetric group, where at each term it only appears one element by column. In order to get a polynomial we need to compensate all the poles that appear in a column:

$$\prod_{j=1}^n \frac{1}{(q^{2k}x_i - q^{-1}y_j)(q^{2k}x_i - qy_j) \dots (q^{2k}x_i - q^{2\ell-1}y_j)}$$

and this is exactly what appears in the numerator of the pre-factor. \square

We can easily count the degree of the partition function:

Lemma 4.2. *The global degree of $\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y})$ is $\delta = \ell n(n-1)$ and the partial degree at each variable x_i (or y_i) is $\delta_i = \ell(n-1)$.*

Proof. The global degree is a very simple computation:

$$\delta = \ell(\ell+1)n^2 - n\ell(\ell-1) - \ell^2n(n-1) - 2\ell n = \ell n(n-1).$$

On the other hand, if we compute blindly the partial degree we get

$$\delta_i = \ell(\ell+1)n - \binom{\ell}{2} - \ell^2(n-1) - 2\ell = \ell(n-1) + \binom{\ell}{2}$$

which is not the expected result. In fact, the extra factor $\binom{\ell}{2}$ disappears when we do a more detailed analysis.

Let x_i be the chosen variable to compute the degree. We want to compute the degree of the determinant in x_i . Notice that there are ℓ columns that depend on x_i , more precisely $\{x_i, \dots, q^{2\ell-2}x_i\}$. Denote each of these columns by $\{v_k(x_i)\}_{k=0}^{\ell-1}$, and notice that they are all similar, i.e. $v_k(x_i) = v_0(q^{2k}x_i)$. We absorb a part of the pre-factor:

$$w_k(x_i) := \prod_{j=1}^n \prod_{p=0}^{\ell} (q^{2k}x_i - q^{2p-1}y_j) v_k(x_i)$$

Now, all the columns are composed by the same polynomials, but evaluated at different points: $x_i, q^2x_i, \dots, q^{2\ell-2}x_i$. We do a Taylor expansion, and we compute the determinant, but we cannot choose the same degree twice, because

⁴ Unfortunately, the letter q is, traditionally used in Macdonald polynomials as well as in the 6-vertex model. And they do not coincide, in fact one is the square of the other. Therefore, we will use (p, t) instead of (q, t) as the two parameters used in the Macdonald polynomials setting, hoping that this will not be confusing.

Theorem 5.1. *The partition function $\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y})$, at the combinatorial point $q = e^{\frac{c\pi i}{2\ell+1}}$, is proportional to the Macdonald polynomial corresponding to the parameters $p = q^2$, $t = q$:*

$$\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y}) \propto P_{Y_{n,\ell}}(\mathbf{x}, \mathbf{y}).$$

Remark 5.2. The coefficients $c(p, t)$ are rational functions on p and t and therefore the limit $(p, t) \rightarrow (q^2, q)$ has to be taken carefully as described below.

This has the important corollary:

Corollary 5.3. *The partition function $\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y})$, in the combinatorial point, is a fully symmetric homogeneous polynomial.*

In what follows, we will sketch the proof of this statement.

5.2. The wheel condition. For convenience, write $\mathbf{z} = \{\mathbf{x}, \mathbf{y}\}$. We define the wheel condition by:

Definition 5.4 (Wheel condition). Let p and t be such that $p^{r-1}t^{k+1} = 1$, for some non-negative integers k and r .

A function $\mathcal{F}(\mathbf{z})$ is said to obey the (r, k) -wheel condition if

$$\mathcal{F}(\mathbf{z}) = 0 \quad \text{if } \frac{z_{i_{\alpha+1}}}{z_{i_{\alpha}}} = tp^{s_{\alpha}} \text{ for all } s_{\alpha} \in \mathbb{N} \text{ such that } \sum_{\alpha=1}^k s_{\alpha} \leq r-1$$

For any choice of $1 \leq i_1 < i_2 < \dots < i_{k+1} \leq 2n$.

In what follows, we consider a choice of rational functions $p = p(u)$ and $t = t(u)$ of a single complex variable u that satisfy $p^{r-1}t^{k+1} = 1$ and $p(u_0) = q^2$ and $t(u_0) = q$ for a particular choice of u_0 .

Let m be the greatest common divisor of $k+1$ and $r-1$. The locus of the equation $p^{r-1}t^{k+1} = 1$ splits in m branches: $p^{\frac{r-1}{m}}t^{\frac{k+1}{m}} = \Omega$, where Ω is a m -th root of the unity. Following [4], we choose to parametrize the branch corresponding to $\Omega = e^{\frac{2\pi i}{m}}$:

$$t(u) = u^{-\frac{r-1}{m}} \omega \quad p(u) = u^{\frac{k+1}{m}}$$

where $\omega = 1$ if $m = 1$, and $\omega = e^{2i\pi/(k+1)}$ otherwise.

Now let $r = \ell$ and $k = 2$. For these the above specialize to

$$\begin{cases} p(u) = u^3, \quad t(u) = u^{-(\ell-1)} & \text{if } \ell \equiv 0 \pmod{3} \\ p(u) = u, \quad t(u) = u^{-\frac{\ell-1}{3}} e^{\frac{2\pi i}{3}} & \text{if } \ell \equiv 1 \pmod{3} \\ p(u) = u^3, \quad t(u) = u^{-(\ell-1)} & \text{if } \ell \equiv 2 \pmod{3}. \end{cases}$$

It is easy to see that with

$$(15) \quad u_0 = \begin{cases} e^{\frac{4\pi i}{3(2\ell+1)}} e^{\frac{2\pi i}{3}} & \text{if } \ell \equiv 0 \pmod{3} \\ e^{\frac{4\pi i}{2\ell+1}} & \text{if } \ell \equiv 1 \pmod{3} \\ e^{\frac{4\pi i}{3(2\ell+1)}} e^{-\frac{2\pi i}{3}} & \text{if } \ell \equiv 2 \pmod{3} \end{cases}$$

we have

$$(16) \quad p(u_0) = q^2 \text{ and } t(u_0) = q.$$

Then the partition function satisfies the wheel condition, or more precisely:

Lemma 5.5. *The partition function $\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y})$ obeys the $(\ell, 2)$ -wheel condition.*

The proof is long, because it should be done case by case, and it is done in Appendix A.

According to the paper [4] by Feigin, Jimbo, Miwa and Mukhin, this vanishing condition, together with the polynomiality and symmetry in the $2n$ variables $\{\mathbf{x}, \mathbf{y}\}$, is enough to prove that it can be expressed as a linear combination of Macdonald polynomials. We remind the reader that $\lambda = (\lambda_1, \dots, \lambda_{2n})$ is an (r, k) -admissible Young diagram if $\lambda_i - \lambda_{i+k} \geq r$.

We recall their statement:

Theorem 5.6 (Feigin, Jimbo, Miwa and Mukhin). *Let p, t be two generic scalars such that $p^{r-1}t^{k+1} = 1$. Let \mathcal{V} be the vector space of symmetric polynomials satisfying the (r, k) -wheel condition. And let \mathcal{M} be the vector space spanned by the Macdonald polynomials, given by (r, k) -admissible Young diagrams and Macdonald parameters p, t .*

Then $\mathcal{V} = \mathcal{M}$.

In our case the statement is reduced to:

Proposition 5.7. *Let p and t be two generic scalars such that $p^{\ell-1}t^3 = 1$, and let $\mathcal{P}(\mathbf{z})$ be an homogeneous fully symmetric polynomial with total degree $\delta = \ell n(n-1)$ obeying to the $(\ell, 2)$ -wheel condition.*

Then, up to a multiplicative constant, this is the Macdonald polynomial $P_{Y_{\ell,n}}(\mathbf{z})$.

Proof. Consider the ring of symmetric polynomials with $2n$ variables. An $(\ell, 2)$ -admissible diagram is such that $\lambda_i - \lambda_{i+2} \geq \ell$, and it is not hard to see that $Y_{n,\ell}$, already defined, is the minimal admissible diagram. If we impose the total degree to be $\delta = \ell n(n-1)$, then the result follows. \square

To use the previous proposition to complete the proof of our main result we have to take care of two important issues. First, the function $\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y})$ is not necessarily symmetric on the $2n$ variables $\{\mathbf{x}, \mathbf{y}\}$. Moreover, we don't use "generic" p and t , and therefore we don't have a guarantee that the theorem by Feigin, Jimbo, Miwa and Mukhin applies here.

5.3. Existence. Given the definition (15) of u_0 , we shall prove that as $u \rightarrow u_0$ we can talk about a suitable renormalization of $P_{Y_{n,\ell}}$ at $u = u_0$ corresponding to the combinatorial point.

The Macdonald polynomials (see for example [13, Chapter VI]), for generic p and t can be decomposed in the monomial basis:

$$P_\mu(\mathbf{z}; p, t) = \sum_{\nu \leq \mu} c_{\mu,\nu}(p, t) m_\nu(\mathbf{z}) \quad \text{with } c_{\mu,\mu}(p, t) = 1$$

where $c_{\mu,\nu}(p, t)$ are rational functions in p and t .

In [4], the authors prove that this is well-defined when we replace p and t by u , in the conditions already described. So we have several rational functions in u , $c_{\mu,\nu}(p(u), t(u))$ for all $\nu \leq \mu$.

Let us see what happens when we approach $u \rightarrow u_0$. Being rational functions, each coefficient will behave like $(u - u_0)^n$ for some power n . Pick the smallest exponent, which is negative or zero (because $c_{\mu,\mu}(p(u), t(u)) = 1$) and call it $-N$. Then the renormalized polynomial

$$(17) \quad \tilde{P}(\mathbf{z}; u) := (u - u_0)^N P_\mu(\mathbf{z}; u)$$

is a regular function around $u \rightarrow u_0$, and the limit can be evaluated by simply replacing u by u_0 .

Proposition 5.8. *If $P_\mu(\mathbf{z}; u)$ satisfies the wheel condition, then the polynomial $\tilde{P}_\mu(\mathbf{z}; u)$ satisfies also the wheel condition.*

Proof. If $u \neq u_0$, both polynomials behave in the same way, because the wheel condition is obtained by evaluation, and one is a multiple of the other one.

When we set $u = u_0$, $P_\mu(\mathbf{z}; u)$ is not necessarily defined, but $\tilde{P}_\mu(\mathbf{z}; u)$ is well-defined (and is non-zero, by construction). Now, the limit $u \rightarrow u_0$ can be seen as evaluating the polynomial in $u = u_0$, and then the two operations commute, and therefore the wheel condition holds. \square

The other properties, like symmetry and degree are obvious by construction.

5.4. Uniqueness. Let V_n be the vector space of homogeneous polynomials defined by:

Definition 5.9. A polynomial $\mathcal{P}(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} = \{x_1, \dots, x_n\}$ and $\mathbf{y} = \{y_1, \dots, y_n\}$, belongs to V_n if and only if it satisfies all the following conditions:

- It is an homogeneous polynomial;
- It is a symmetric polynomial on the variables \mathbf{x} , idem on the variables \mathbf{y} ;
- It is stable by exchanging $\mathbf{x} \leftrightarrow \mathbf{y}$;
- It has a total degree $\delta = \ell n(n-1)$;
- It has a partial degree $\delta = \ell(n-1)$;
- It obeys the $(\ell, 2)$ -wheel condition.

Then the next lemma holds:

Lemma 5.10 (Uniqueness). *The vector space V_n is at most one dimensional.*

The proof shall be postponed to Appendix B.

Now the proof of the main theorem (i.e. Theorem 5.1) is trivial.

Proof. Note that the definition of V_n is such that $\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y}) \in V_n$, as well as $P_{Y_{n,\ell}}(\mathbf{x}, \mathbf{y})$. But by Lemma 5.10, V_n is at most one dimensional, and therefore $\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y}) \propto P_{Y_{n,\ell}}(\mathbf{x}, \mathbf{y})$. \square

5.5. Proportionality constant. In order to finish the proof we shall prove that

Proposition 5.11. *The constant $\gamma_{n,\ell}$ defined by $\mathcal{Z}_{n,\ell} = \gamma_{n,\ell} P_{Y_{n,\ell}}$ is given by:*

$$\gamma_{n,\ell} = (-1)^n \binom{\ell}{2} q^{2(n-1)\ell(\ell-1)} ([\ell]!)^n ,$$

which is non-zero.

Where

$$[\ell]! = [\ell][\ell-1] \dots [1] \quad [a] = \frac{q^a - q^{-a}}{q - q^{-1}} .$$

Proof. The polynomial $P_{Y_{n,\ell}}(\mathbf{x}, \mathbf{y})$ can be written as:

$$P_{Y_{n,\ell}} = m_{Y_{n,\ell}} + \sum_{\mu < Y_{n,\ell}} c_\mu(p, t) m_\mu .$$

So, the constant $\gamma_{n,\ell}$ is just the leading coefficient of the partition function. This computation will be postponed to the Appendix A.5, where, to be more precise, we compute the coefficient of $\prod_{i=1}^n (x_i y_i)^{\ell(i-1)}$. \square

6. FINAL REMARKS

6.1. Combinatorial interpretation. The 6-vertex configurations are in bijection with several combinatorial objects, like the Alternating Sign Matrices (ASM) and Fully Packed Loops (FPL). For example, using the partition function of the 6-vertex model, Kuperberg (see [12]) computed the number of Alternating Sign Matrices.

A canonical generalization of the ASM was given by Berheng and Knight in 2007 [1]: We have a $n \times n$ matrix A with entries $A_{i,j} \in \{-\ell, \dots, -1, 0, 1, \dots, \ell\}$. The sum of each column or row is ℓ . And any partial sum (the sum of the first or last r entries of each row or column) is nonnegative.

In fact, the bijection is trivial, i.e., it is exactly the same of the normal case. Read each column from the bottom to the top, the value of the edge goes from β to η , then the corresponding entry of the higher spin ASM is $\eta - \beta$. The Higher Spin 6-vertex model guarantee that we get the same result if we read a row from the right to the left. We can easily prove that this is a bijection.

Unfortunately, the weights of our model seem very unnatural.

In the combinatorial point, the partition function is a Macdonald polynomial. In the case $\ell = 1$, when we replace all variables by 1, we get the famous sequence that counts several things: Alternating Sign Matrices, Totally Symmetric Self-Complementary Plane Partitions.

Therefore it will be very interesting to find some combinatorial meaning to the values obtained in the homogeneous limit, i.e. $x_i = y_i = 1$ for all i .

6.2. The wheel condition. The fact that the partition function obeys the wheel condition seems to be a consequence of its determinantal form. This should lead us to a more systematical study about the wheel condition and determinants.

And, can we get the same wheel condition but for higher degree, or this process always gives the smallest possible degree?

In other direction, other generalizations of the 6-vertex model are known. The most interesting one is that instead of choosing Higher Spin representations of sl_2 , we can use different algebras, as for example so_n , which appeared in a paper of Dow and Foda [3].

6.3. Symmetry. It is very intriguing that we start with a function that is not necessarily symmetric by S_{2n} (but only $S_n \times S_n$). What is so special about this determinant that gives us this extra symmetry? It is expectable that the symmetry is coming from the physics of the model, that is, there exists some mechanism (like the Yang-Baxter equation) which, for this very special value of q , tells us that we can exchange one row with one column.

Another idea for explaining the symmetry comes from Stroganov's article [17].

6.4. Kadomtsev-Petviashvili tau functions. Our method to prove the wheel condition for the partition function at the combinatorial point has a Grassmannian manifold interpretation, and this approach is intimately connected with the fact that the six vertex model with domain wall boundary conditions can be seen as a τ -function of the KP hierarchy where the spectral parameters \mathbf{x} and \mathbf{y} are playing the role of Miwa variables. This was first observed by Foda, Wheeler and Zuparic [5], based on a fermionic vacuum expectation value representation of the partition function. In his survey paper [19], Takasaki extends this and other related results by showing how partition functions of certain 2D solvable models and scalar products of Bethe vectors from integrable spin chain models can also be written as KP τ -functions.

APPENDIX A. PROOF OF THE WHEEL CONDITION

We prove the wheel condition based on expressing the determinant appearing in the definition of the partition function in the form

$$(18) \quad \det(\langle v_i, v_j^* \rangle)_{i,j=1}^{\ell n} = \langle v_1 \wedge \cdots \wedge v_{\ell n}, v_1^* \wedge \cdots \wedge v_{\ell n}^* \rangle$$

for a collection of vectors $v_1, \dots, v_{\ell n} \in V$ depending on \mathbf{x} and linear functionals $v_1^*, \dots, v_{\ell n}^* \in V^*$ depending on \mathbf{y} for a suitably chosen vector space V and dual space V^* equipped with the natural pairing $\langle \cdot, \cdot \rangle: V \times V^* \rightarrow \mathbb{C}$.

It will be shown that the vectors $\{v_i\}_{i=1}^{\ell n}$ are linearly dependent if the XXX wheel condition holds and, similarly, the YYY wheel condition implies that $v_1^* \wedge \cdots \wedge v_{\ell n}^* = 0$. The mixed wheel conditions XXY and XYX result from the interplay between the linear subspaces spanned by the v 's and v^* 's respectively.

A.1. A scalar product perspective. We will use indices i and j from 1 to n to label the $\ell \times \ell$ blocks and indices α and β from 0 to $(\ell - 1)$ to label the entries in a specific block. For combinations of the form (i, α) we use a and b .

Consider our partition function:

$$\begin{aligned} \mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y}) &= \frac{\prod_{i,j} \prod_{p=0}^{\ell} \prod_{k=0}^{\ell-1} (q^{2k} x_i - q^{2p-1} y_j)}{\Delta(\bar{\mathbf{x}}) \Delta(\bar{\mathbf{y}})} \det \frac{1}{(q\bar{x}_a - \bar{y}_b)(q^{-1}\bar{x}_a - \bar{y}_b)} \\ &= \frac{1}{\Delta(\bar{\mathbf{x}}) \Delta(\bar{\mathbf{y}})} \det \tilde{P}_a(\bar{y}_b) \\ &= \frac{1}{\prod_a \pi(\bar{y}_a) \Delta(\bar{\mathbf{x}}) \Delta(\bar{\mathbf{y}})} \det P_a(\bar{y}_b) \end{aligned}$$

where

$$\begin{aligned} \tilde{P}_{(i,\alpha)}(z) &= \prod_{j \neq i} \prod_{\beta=0}^{\ell} (q^{2\beta-1} x_j - z) \prod_{\substack{0 \leq \beta < \ell \\ \beta \neq \alpha, \alpha+1}} (q^{2\beta-1} x_i - z) \\ P_a(z) &= \prod_{b \neq a} (q\bar{x}_b - z)(q^{-1}\bar{x}_b - z) = \pi(z) \tilde{P}_a(z) \\ \pi(z) &= \prod_j \prod_{\beta=1}^{\ell-1} (q^{2\beta-1} x_j - z) \end{aligned}$$

Notice that the degree of $\tilde{P}_a(z)$ is $(\ell + 1)n - 2$ and the degree of $P_a(z)$ is $2\ell n - 2$. For what follows we will use the P_a , and we will consider that $\{\bar{\mathbf{x}}, \bar{\mathbf{y}}\}$ are $2\ell n$ independent variables. We should return to \tilde{P}_a later.

Define the scalar product by:

$$\langle f, g \rangle = \frac{1}{2\pi i} \oint_{S^1} f(z) g(z) dz$$

And let

$$\chi_j = \frac{1}{z - \bar{y}_j}$$

then

$$\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y}) = \frac{1}{\prod_a \pi(\bar{y}_a) \Delta(\bar{\mathbf{x}}) \Delta(\bar{\mathbf{y}})} \det \langle P_a, \chi_b \rangle$$

Write $P_a(z) = \sum_{b=1}^{2\ell n-1} p_{ab} z^{b-1}$. And define the rectangular matrix $\mathcal{P} = [p_{ab}]$.

A.2. A new polynomial and a basis transformation. We introduce now a complementary polynomial:

$$Q_{\bar{x},a}(z) = \prod_{b \neq a} (z - \bar{x}_b) = \sum_{b=1}^{\ell n} q_{ab} z^{b-1}$$

and of course we define the square matrix $\mathcal{Q}_{\bar{x}} = [q_{ab}]$.

Let $\mathcal{V}_{\bar{x}}$ be the Vandermonde matrix:

$$\mathcal{V}_{\bar{x}} = \begin{bmatrix} 1 & \dots & 1 \\ \bar{x}_1 & \dots & \bar{x}_{\ell n} \\ \vdots & & \vdots \\ \bar{x}_1^{\ell n-1} & \dots & \bar{x}_{\ell n}^{\ell n-1} \end{bmatrix}$$

It is then trivial that $Q_{\bar{x},a}(\bar{x}_c) = \sum_b q_{ab} (\mathcal{V}_{\bar{x}})_{bc} = \delta_{ac} q_a$, where $q_a = \prod_{c \neq a} (\bar{x}_a - \bar{x}_c)$. Then we can compute the inverse matrix:

$$\mathcal{Q}_{\bar{x}}^{-1} = \mathcal{V}_{\bar{x}} \begin{bmatrix} q_1^{-1} & & \\ & \ddots & \\ & & q_{\ell n}^{-1} \end{bmatrix}$$

and, consequently, $\det \mathcal{Q}_{\bar{x}} = (-1)^{\binom{\ell n}{2}} \Delta(\bar{x})$.

Let $R_a(z) = \sum_{b=1}^{2\ell n-1} r_{ab} z^{b-1}$ be a new set of polynomials, and let $\mathcal{R} = [r_{ab}]$ be the corresponding matrix. This new polynomial is defined by $\mathcal{R} = \mathcal{Q}_{\bar{x}}^{-1} \mathcal{P}$, i.e.

$$R_a(z) = \sum_b \bar{x}_b^{a-1} q_b^{-1} P_b(z)$$

which simplifies the partition function:

$$\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y}) = \frac{(-1)^{\binom{\ell n}{2}}}{\prod_a \pi(\bar{y}_a) \Delta(\bar{\mathbf{y}})} \det \langle R_a, \chi_b \rangle$$

It is not obvious that $R_a(z)$ is a polynomial in \bar{x} . Indeed, we will need to compute it in order to assert that.

A.3. Computing the new polynomials $R_a(z)$. We rewrite the polynomial $R_a(z)$, for generic z , using contour integrals:

$$R_a(z) = \prod_c (q\bar{x}_c - z)(q^{-1}\bar{x}_c - z) \left(\frac{1}{2\pi i} \oint \frac{1}{(\xi - qz)(\xi - q^{-1}z)} \frac{\xi^{a-1}}{\prod_c (\xi - \bar{x}_c)} d\xi \right)$$

where the contour of integration encircles all \bar{x}_c but not $q^{\pm 1}z$.

There is no pole at $\xi = \infty$, therefore we can invert the integration:

$$R_a(z) = \frac{1}{(q - q^{-1})z} \prod_c (qz - \bar{x}_c) (zq^{-1})^{a-1} - \frac{1}{(q - q^{-1})z} \prod_c (q^{-1}z - \bar{x}_c) (zq)^{a-1}$$

Let $T(z) = \prod_c (z - \bar{x}_c)$, we can reinterpret this as⁵:

$$R_a(z) = \frac{T(qz)(q^{-1}z)^{a-1} - T(q^{-1}z)(qz)^{a-1}}{(q - q^{-1})z}$$

These polynomials generate a specific vector subspace:

$$\begin{aligned} \mathcal{S} &= \text{span} \{ R_a(z) : 1 \leq a \leq \ell n \} \\ &= \left\{ \frac{T(qz)f(q^{-1}z) - T(q^{-1}z)f(qz)}{(q - q^{-1})z} : f(z) \text{ is a polynomial of degree } < \ell n \right\} \end{aligned}$$

⁵In fact this can be seen as the q -difference of $T(z)z^{-a}$.

If we think now as $\bar{x}_{(i,\alpha)} = q^{2\alpha}x_i$, we notice that there are some common factors between $T(qz)$ and $T(q^{-1}z)$ which combine exactly to $\pi(z)$. Then we can absorb the factor $\prod_a \pi(\bar{y}_a)$ in the definition:

$$\tilde{\mathcal{S}} = \left\{ \frac{\tilde{T}(q^\ell z)f(q^{-1}z) - \tilde{T}(q^{-\ell}z)f(qz)}{(q - q^{-1})z} : f(z) \text{ is a polynomial of degree } < \ell n \right\}$$

where $\tilde{T}(z) = \prod_i (z - q^{\ell-1}x_i)$.

And using $f(z) = z^{a-1}$, we can form a basis $\tilde{R}_a(z)$.

A.4. Second basis transformation. We can do the same basis transformation for the χ_a 's:

$$\chi_a = \frac{1}{z - \bar{y}_a} = \frac{\prod_{b \neq a} (z - \bar{y}_b)}{\prod_b (z - \bar{y}_b)} = \frac{Q_{\bar{y},a}}{\prod_b (z - \bar{y}_b)} = \sum_b \frac{q_{ab} z^{b-1}}{\prod_c (z - \bar{y}_c)}$$

Then the partition function can be rewritten as:

$$\begin{aligned} \mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y}) &= \frac{(-1)^{\binom{\ell n}{2}}}{\prod_a \pi(\bar{y}_a) \Delta(\bar{\mathbf{y}})} \det \langle R_a, \chi_b \rangle \\ &= \frac{(-1)^{\binom{\ell n}{2}} \det Q_{\bar{\mathbf{y}}}}{\prod_a \pi(\bar{y}_a) \Delta(\bar{\mathbf{y}})} \det \left\langle R_a, \frac{z^{b-1}}{\prod_c (z - \bar{y}_c)} \right\rangle \\ &= \frac{1}{\prod_a \pi(\bar{y}_a)} \det \langle R_a, \theta_b \rangle \end{aligned}$$

where

$$\theta_b = \frac{z^{b-1}}{\prod_c (z - \bar{y}_c)}.$$

Furthermore, we could absorb the factor $\pi(\bar{y}_a)$ which amounts to simplifying the expression to $\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y}) = \det \langle \tilde{R}_a, \theta_b \rangle$.

A.5. Leading coefficient. At this stage, computing the leading coefficient of the partition function, necessary to the proof of Proposition 5.11, is relatively easy. Take the following expression

$$\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y}) = \frac{(-1)^{\binom{\ell n}{2}}}{\Delta(\bar{\mathbf{y}})} \det \langle \tilde{R}_a, \chi_b \rangle$$

and use the same trick as in A.4 for the variables $\{y_1, q^2 y_1, \dots, q^{2\ell-2} y_1\}$:

$$\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y}) = \frac{(-1)^{\binom{\ell(n-1)}{2} + \ell(n-1)}}{\prod_{i=2}^n \prod_{k,p} (q^{2p} y_i - q^{2k} y_1) \Delta_{\hat{1}}(\bar{\mathbf{y}})} \det \langle \tilde{R}_a, \chi'_b \rangle$$

where $\Delta_{\hat{1}}(\bar{\mathbf{y}})$ and χ'_b are obtained from $\Delta(\bar{\mathbf{y}})$ and χ_b after some small modifications: we remove all terms depending in y_1 from the Vandermonde; and the first ℓ functions χ_b are replaced by $\frac{z^{b-1}}{\prod_{i=0}^{\ell-1} (z - q^{2i} y_1)}$.

Set $y_1 = x_1 = 0$. Then we have

$$\tilde{R}_a(z) = \frac{q^{\ell-a+1} z^a \prod_{i=2}^n (q^\ell z - q^{\ell-1} x_i) - q^{-\ell+a-1} z^a \prod_{i=2}^n (q^{-\ell} z - q^{\ell-1} x_i)}{(q - q^{-1})z}.$$

and $\chi'_b = z^{-(1+\ell-b)}$ for $b \leq \ell$, then the scalar product $\langle \tilde{R}_a, \chi'_b \rangle$ vanishes whenever $a > 1 + \ell - b$. I.e. the matrix $\langle \tilde{R}_a, \chi'_b \rangle$ has ℓ rows that are practically zero, except for a triangle on the top. In this case, we can remove these rows from the determinant, together with the first ℓ columns.

In this process we get the factor:

$$\begin{aligned} (-1)^{\binom{\ell}{2}} \prod_{a=1}^{\ell} \langle \tilde{R}_a, z^{-a} \rangle &= (-1)^{\binom{\ell}{2}} \prod_{a=1}^{\ell} \frac{1}{2\pi i} \oint \tilde{R}_a(z) z^{-a} dz \\ &= (-1)^{\binom{\ell}{2} + \ell(n-1)} q^{(n-1)\ell(\ell-1)} [\ell]! \prod_{i=2}^n x_i^{\ell}. \end{aligned}$$

The remaining part of the determinant is very similar to the original one, except that \tilde{R}_a starts with $a = \ell + 1$. Notice that

$$\tilde{R}_{\ell+a}(z) \Big|_{x_1=0} = z^{\ell+1} \tilde{R}_{a,\hat{1}}(z)$$

where $\tilde{R}_{a,\hat{1}}$ is the version of \tilde{R}_a with one less variable: x_1 . In the scalar product this factorizes: $\langle \tilde{R}_{\ell+a} \Big|_{x_1=0}, \chi_b \rangle = (\bar{y}_b)^{\ell+1} \langle \tilde{R}_{a,\hat{1}}, \chi_b \rangle$.

Joining all the contributions, we get

$$\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y}) \Big|_{x_1=y_1=0} = (-1)^{\binom{\ell}{2}} q^{2(n-1)\ell(\ell-1)} [\ell]! \left(\prod_{i=2}^n y_i^{\ell} x_i^{\ell} \right) \mathcal{Z}_{n-1,\ell}(\mathbf{x}_{\hat{1}}, \mathbf{y}_{\hat{1}}),$$

which can be used recursively to compute the coefficient of $\prod_{i=1}^n (x_i y_i)^{\ell(i-1)}$:

$$\gamma_{n,\ell} = (-1)^n q^{n(n-1)\ell(\ell-1)} ([\ell]!)^n.$$

A.6. The XXX wheel condition. Now, that we have a different perspective of our expression, we can use it in order to prove the wheel condition. We start by proving the XXX wheel condition. From now on we assume that $q^{2\ell+1} = 1$

Pick three elements of \mathbf{x} , say x_1, x_2 and x_3 (notice that the partition function is symmetric, therefore it does not matter which elements we choose) and set $x_3 = q^{1+2s_2} x_2 = q^{2+2s_1+2s_2} x_1$, where s_1 and s_2 are such that $s_1 + s_2 \leq \ell - 1$.

The XXX wheel condition is a simple corollary of the following lemma:

Lemma A.1. *If we set $x_3 = q^{1+2s_2} x_2 = q^{2+2s_1+2s_2} x_1$, the dimension of the vector space \mathcal{S} is less than ℓn .*

Proof. In other words, there is a polynomial $f(z)$ of degree less than ℓn , such that

$$\frac{T(qz)f(q^{-1}z) - T(q^{-1}z)f(qz)}{(q - q^{-1})z} = 0$$

This means that:

$$\frac{f(qz)}{f(q^{-1}z)} = \frac{T(qz)}{T(q^{-1}z)}$$

Suppose that $T(z)$ and $f(z)$ share the same behavior for x_i with $i > 3$, i.e.,

$$\begin{aligned} T(z) &= \hat{T}(z) \prod_{i=4}^n \prod_{\alpha=0}^{\ell-1} (z - q^{2\alpha} x_i) \\ f(z) &= \hat{f}(z) \prod_{i=4}^n \prod_{\alpha=0}^{\ell-1} (z - q^{2\alpha} x_i) \end{aligned}$$

We want to find a polynomial $\hat{f}(z)$ with degree smaller than 3ℓ such that:

$$\frac{\hat{f}(qz)}{\hat{f}(q^{-1}z)} = \frac{\hat{T}(qz)}{\hat{T}(q^{-1}z)}$$

Now, set $x_3 = q^{1+2s_2}x_2 = q^{2+2s_1+2s_2}x_1$, and look at the quotient on the RHS:

$$\begin{aligned} \frac{\hat{T}(qz)}{\hat{T}(q^{-1}z)} &= \prod_{\alpha=0}^{\ell-1} \frac{(qz - q^{2\alpha}x_1)}{(q^{-1}z - q^{2\alpha}x_1)} \frac{(qz - q^{2\alpha}x_2)}{(q^{-1}z - q^{2\alpha}x_2)} \frac{(qz - q^{2\alpha}x_3)}{(q^{-1}z - q^{2\alpha}x_3)} \\ &= q^{6\ell} \prod_{\alpha=0}^{\ell-1} \frac{(z - q^{2\alpha-1}x_1)}{(z - q^{2\alpha+1}x_1)} \frac{(z - q^{2\alpha-1}x_2)}{(z - q^{2\alpha+1}x_2)} \frac{(z - q^{2\alpha-1}x_3)}{(z - q^{2\alpha+1}x_3)} \\ &= q^{6\ell} \frac{(z - q^{-1}x_1)}{(z - q^{2\ell-1}x_1)} \frac{(z - q^{2s_1}x_1)}{(z - q^{2s_1-1}x_1)} \frac{(z - q^{2s_1+2s_2+1}x_1)}{(z - q^{2s_1+2s_2}x_1)} \end{aligned}$$

Call tail of x the sequence $\{x, q^2x, \dots, q^{2\ell-2}x\}$. Using the fact that $q^{2\ell+1} = 1$, let us represent this in a graphical way, for the example $\ell = 6$, $s_1 = 2$ and $s_2 = 1$ (then $q^{13} = 1$):

$$\begin{array}{cccccccccccccccc} & 1 & q & q^2 & q^3 & q^4 & q^5 & q^6 & q^7 & q^8 & q^9 & q^{10} & q^{11} & q^{12} \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \text{tails of } 1 : & \bullet & + & \bullet & + & \bullet & + & \bullet & + & \bullet & + & \bullet & + & \oplus_1 \\ & & - & & - & & - & & - & & - & & - & \ominus_1 \\ \text{tails of } q^5 : & \bullet & + & \bullet & & \oplus_2 & \bullet & + & \bullet & + & \bullet & + & \bullet & + \\ & & - & & \ominus_2 & & - & & - & & - & & - & \\ \text{tails of } q^8 : & + & \bullet & + & \bullet & + & \bullet & & \oplus_3 & \bullet & + & \bullet & + & \bullet \\ & - & & - & & - & & \ominus_3 & & - & & - & & \end{array}$$

In the figure above, the black dots mark the zeros of $\prod_{\alpha} (z - q^{2\alpha}x_1)$, and the pluses or minuses mark the corresponding zeros or poles in our expression. Notice that all terms in each pair of tails cancel except for the first in the “positive” tail and the last one in the negative tail. Then the final result is

$$\frac{\hat{T}(qz)}{\hat{T}(q^{-1}z)} = \frac{\oplus_1 \oplus_2 \oplus_3}{\ominus_1 \ominus_2 \ominus_3}$$

We try now to solve the inverse problem. We have the final quotient, composed by three \oplus_i ’s and three \ominus_i ’s⁶, and knowing that the $-$ are moved by two to the right, how can we obtain it. For example, the case of $\hat{T}(z)$ corresponds to $\{\oplus_1 \rightarrow \ominus_1, \oplus_2 \rightarrow \ominus_2, \oplus_3 \rightarrow \ominus_3\}$.

There are of course $3! = 6$ different choices. But if we impose that the number of pluses is less than 3ℓ , there is only one which does the job, the : $\{\oplus_1 \rightarrow \ominus_2, \oplus_2 \rightarrow \ominus_3, \oplus_3 \rightarrow \ominus_1\}$, see the example:

$$\begin{array}{cccccccccccccccc} & 1 & q & q^2 & q^3 & q^4 & q^5 & q^6 & q^7 & q^8 & q^9 & q^{10} & q^{11} & q^{12} \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \text{tails of } 1 : & \bullet & + & \bullet & & & & & & & & & & \oplus_1 \\ & & - & & \ominus_2 & & & & & & & & & \\ \text{tails of } q^5 : & & & & & \oplus_2 & \bullet & & & & & & & \\ & & & & & & \ominus_3 & & & & & & & \\ \text{tails of } q^8 : & & & & & & & \oplus_3 & \bullet & + & \bullet & & & \ominus_1 \\ & & & & & & & & & - & & & & \end{array}$$

⁶In fact it is important that $(\oplus_1, \oplus_2, \oplus_3) = (q^{-1}, q^{2s_1}, q^{1+2s_1+2s_2})$ and $(\ominus_1, \ominus_2, \ominus_3) = (q^{-2}, q^{2s_1-1}, q^{2s_1+2s_2})$.

So if we set $\hat{f}(z) = (z - x_1)(z - q^2x_1)(z - q^5x_1)(z - q^8x_1)(z - q^{10}x_1)$, we obtain:

$$\frac{\hat{f}(qz)}{\hat{f}(q^{-1}z)} = q^{10} \frac{(z - q^{-1}x_1)(z - q^4x_1)(z - q^7x_1)}{(z - q^3x_1)(z - q^6x_1)(z - q^{11}x_1)}$$

Let us be more precise, recall that $\hat{T}(z) = \prod_{\alpha=0}^{\ell-1} (z - q^{2\alpha}x_1)(z - q^{1+2s_1+2\alpha}x_1) \times (z - q^{2(1+s_1+s_2+\alpha)}x_1)$. And we have exactly 3ℓ terms, which explains the factor $q^{6\ell}$ that appears. Now, construct $\hat{f}(z)$:

$$\hat{f}(z) = \prod_{\alpha=0}^{s_1-1} (z - q^{2\alpha}x_1) \prod_{\alpha=0}^{s_2-1} (z - q^{1+2s_1+2\alpha}x_1) \prod_{\alpha=0}^{\ell-s_1-s_2-2} (z - q^{2(1+s_1+s_2+\alpha)}x_1)$$

which has exactly $\ell - 1$ terms (which is smaller than 3ℓ as wanted). The factor that appears is $q^{2\ell-2}$, but $q^{2\ell-2} = q^{6\ell}$, then

$$\frac{\hat{f}(qz)}{\hat{f}(q^{-1}z)} = \frac{\hat{T}(qz)}{\hat{T}(q^{-1}z)}$$

holds and the vector space \mathcal{S} is at most $\ell n - 1$ dimensional. \square

Corollary A.2. *The partition function satisfies the XXX wheel condition.*

Proof. This is obvious. We know that $\mathcal{Z}_n(\mathbf{x}, \mathbf{y}) = \frac{1}{\prod_a \pi(\bar{y}_a)} \det \langle R_a, \theta_j \rangle$, and it is given that $\pi(\bar{y}_a)$ do not vanish. So if R_a span a vector space whose dimension is smaller than the size of the matrix, the determinant vanishes. \square

A.7. The YYY wheel condition. By the symmetry of the problem we know that the YYY wheel condition also holds. Anyway, we present here a proof of the YYY wheel condition without using the symmetry.

Instead we will use the following lemma:

Lemma A.3. *If $y_3 = q^{1+2s_2}y_2 = q^{2+2s_1+2s_2}y_1$, with $s_1 + s_2 \leq \ell - 1$, then it is possible to find a subset of $(2\ell + 1)$ variables $\{\tilde{y}_0, \dots, \tilde{y}_{2\ell}\}$ in*

$$\bar{\mathbf{y}} = \{y_1, q^2y_1, \dots, q^{2\ell-2}y_1, y_2, q^2y_2, \dots, q^{2\ell-2}y_2, y_3, q^2y_3, \dots, q^{2\ell-2}y_3\},$$

such that $\tilde{y}_i = q^{2i}\tilde{y}_0$.

By the symmetry in \mathbf{y} , this also holds for any three variables in \mathbf{y} .

Proof. For simplicity, let $y_1 = 1$. Notice that the set $\{1, q^2, q^4, \dots, q^{4\ell}\}$ is the same as $\{1, q, q^2, \dots, q^{2\ell}\}$, by the fact that $q^{2\ell+1} = 1$.

Then the lemma is equivalent to the statement that the tails of $1, q^{1+2s_1}$ and $q^{2+2s_1+2s_2}$ cover the full set $\{1, q, q^2, \dots, q^{2\ell}\}$.

For example, let $\ell = 6$, $s_1 = 2$ and $s_2 = 1$. Build a table, with the tails of $1, q^5$ and q^8 :

	1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}	q^{12}
	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet
tail of 1 :	\circ		\circ		\circ		\circ		\circ		\circ		
tail of q^5 :	\circ		\circ			\circ		\circ		\circ		\circ	
tail of q^8 :		\circ		\circ		\circ			\circ		\circ		\circ

Therefore, when we look at some power of q , we are sure that it appears in, at least, one of the tails.

Now, we should prove that this always happens. The tail of 1 covers all even powers of q (the red dots), but $q^{2\ell}$, which is covered by the tail of $q^{2+2s_1+2s_2}$ since $2 + 2s_1 + 2s_2 \leq 2\ell$.

If $s_1 = 0$, then the tail of $q^{1+2s_1} = q$ covers all odd powers, otherwise it covers only the odd powers at least q^{1+2s_1} . $q^{2+2s_1+2s_2}$ is even, but it reaches 2ℓ and go around becoming odd, the last value is $q^{2+2s_1+2s_2+2\ell-2} = q^{2s_1+2s_2-1}$. In the worst case scenario, $s_2 = 0$, it goes up to q^{2s_1-1} covering the rest of the odd powers.

We are done. \square

We need the following proposition:

Proposition A.4. *Let \mathcal{M} be an square matrix of odd size $(2n+1) \times (2n+1)$, with shape:*

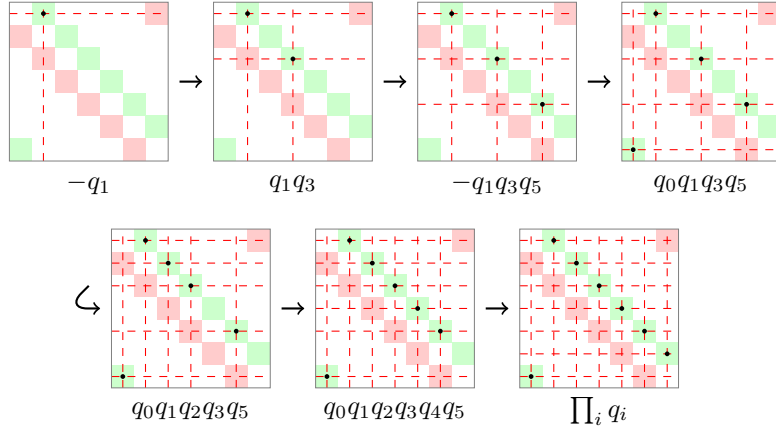
$$\mathcal{M} = \begin{bmatrix} 0 & q_1 & 0 & 0 & \dots & 0 & p_0 \\ p_1 & 0 & q_2 & 0 & \dots & 0 & 0 \\ 0 & p_2 & 0 & q_3 & \dots & 0 & 0 \\ \vdots & & \ddots & & \ddots & \vdots & \vdots \\ \vdots & & & \ddots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 & q_{2n} \\ q_0 & 0 & 0 & 0 & \dots & p_{2n} & 0 \end{bmatrix}$$

Then $\det \mathcal{M} = \prod_{i=0}^{2n} q_i + \prod_{i=0}^{2n} p_i$.

Proof. In order to prove this, we work out a simple example. Set $n = 3$, then our matrix is of the form:

where the entries missing vanish.

In the first row, we can choose q_1 or p_0 . If we chose q_1 , we remove the first row and the second column:



As we can see, if we pick q_1 , we will be forced to pick q_3, q_5 and so on up to q_{2n-1} . And we get always a minus sign, at this point we have $(-1)^n$. Now we are forced to get q_0 , which is in the first column and last row (notice that we had removed n

rows already), then it gets a sign $(-1)^n$. This will force q_2 , and so on up to q_{2n} , without getting any signs.

We can do the same game with the p_i starting with p_0 and we are done. Notice that the fact that it is an odd matrix is very important. We do not need it, but there also is an even version of this proposition. \square

Now, we want to prove that our matrix $\langle R_a, \chi_b \rangle$ is singular. We will prove that there are $2\ell + 1$ rows that are linearly dependent.

Lemma A.5. *Consider the $2\ell + 1$ rows corresponding to $\chi_i = \frac{1}{z - q^i y}$, where $0 \leq i \leq 2\ell$, whose existence of $\tilde{\mathbf{y}} = \{q^i y\}_{i=0}^{2\ell}$ was proved in Lemma A.3. Then the matrix $2n\ell \times (2\ell + 1)$ $\langle R_a, \chi_i \rangle$ has rank at most 2ℓ .*

Proof. This is not hard to prove. Let $p(z)$ be an arbitrary polynomial of degree less than ℓn , then we obtain one vector in \mathcal{S} by:

$$R(z) = \frac{T(qz)p(q^{-1}z) - T(q^{-1}z)p(qz)}{(q - q^{-1})z}$$

Then, when we evaluate this vector in the set $\tilde{\mathbf{y}}$ is of the form:

$$\begin{bmatrix} R(y) \\ R(qy) \\ R(q^2y) \\ \vdots \\ R(q^{2\ell}y) \end{bmatrix} = \mathcal{M} \begin{bmatrix} 0 & -T(q^{2\ell}y) & 0 & \dots & T(qy) \\ T(q^2y) & 0 & -T(y) & \dots & 0 \\ 0 & T(q^3y) & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ -T(q^{2\ell-1}y) & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} p(y) \\ p(qy) \\ p(q^2y) \\ \vdots \\ p(q^{2\ell}y) \end{bmatrix}$$

with

$$\mathcal{M} = \left[\frac{1}{q - q^{-1}} \frac{1}{q^i y} \delta_{ij} \right]_{i,j=0}^{2\ell}.$$

Call $\mathcal{T} = [t_{ij}]_{i,j=0}^{2\ell}$ the matrix in the middle, by Proposition A.4, we know that the determinant of $\mathcal{T} = (-1)^{2\ell+1} \prod_i T(q^i y) + \prod_i T(q^i y) = 0$.

This means that the $2\ell + 1$ linear functionals corresponding to evaluations at $y, qy, \dots, q^{2\ell}y$ are linearly independent on \mathcal{S} . Therefore the original determinant has $2\ell + 1$ columns that are linearly independent. \square

The YYY wheel condition follows.

A.8. The XXY wheel condition. For proving the XXY wheel condition, we will use the expression

$$\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y}) = \frac{(-1)^{\binom{\ell n}{2}}}{\Delta(\tilde{\mathbf{y}})} \det \langle \tilde{R}_a, \chi_b \rangle \quad \text{where } \tilde{R}_a \in \tilde{\mathcal{S}}$$

the term $\Delta(\tilde{\mathbf{y}})$ does not present any problem when we set $y_1 = q^{1+2s_2}x_2 = q^{2+2s_1+2s_2}x_1$, we will use $x_1 = 1$, for simplicity.

For any polynomial $p(z)$ we have

$$\tilde{R}(z) = \frac{\tilde{T}(q^\ell z)p(q^{-1}z) - \tilde{T}(q^{-\ell}z)p(qz)}{(q - q^{-1})z}$$

where

$$\begin{aligned} \tilde{T}(q^\ell z) &= q^{-1}(z - q^{-1})(z - q^{2s_1}) \prod_{i>2} (q^\ell z - q^{\ell-1}x_i) \\ \tilde{T}(q^{-\ell}z) &= q(z - q^{-2})(z - q^{2s_1-1}) \prod_{i>2} (q^\ell z - q^{\ell-1}x_i) \end{aligned}$$

Let $p_0(z) = \prod_{\alpha=0}^{s_1-1} (z - q^{2\alpha})$, which has degree s_1 . The space of polynomials of degree less than ℓn and divisible by $p_0(z)$ is $(\ell n - s_1)$ -dimensional.

Then it is possible to create a basis for the space $\tilde{\mathcal{S}}$ composed by $(\ell n - s_1)$ elements of the form:

$$\tilde{R}_a(z) = \frac{\tilde{T}(q^\ell z)p_a(q^{-1}z)p_0(q^{-1}z) - \tilde{T}(q^{-\ell}z)p_a(qz)p_0(qz)}{(q - q^{-1})z},$$

where p_a is some polynomial of degree less than $(\ell n - s_1)$, supplemented with s_1 additional linearly independent elements generated by polynomials not divisible by p_0 .

We will need the following proposition:

Proposition A.6. *Given s_1 , and writing $y_1 = q^{2+2s_1+2s_2}$, then $\{q^{-1}, q, \dots, q^{2s_1-1}\}$, call it $\tilde{\mathbf{y}}$, always appear in $\{q^{2\alpha}y_1\}_{\alpha=0}^{\ell-1}$, independently of s_2 , besides the fact that $s_1 + s_2 \leq \ell - 1$.*

Proof. This is obvious, let us study the two extrema. If $s_2 = 0$, we have:

$$\{q^{2+2s_1}, q^{4+2s_1}, \dots, q^{2\ell-2+2+2s_1}\} = \{q^{2+2s_1}, q^{4+2s_1}, \dots, q^{2s_1-1}\}$$

notice that $2 + 2s_1 \leq 2\ell$.

In the other extreme, $s_1 + s_2 = \ell - 1$:

$$\{q^{2\ell}, q^{2\ell+2}, \dots, q^{4\ell-2}\} = \{q^{-1}, q, \dots, q^{2\ell-3}\}$$

notice that $2s_1 - 1 \leq 2\ell - 3$. □

Notice that $\tilde{\mathbf{y}}$ has $(s_1 + 1)$ elements.

Lemma A.7. *The XXY wheel condition holds.*

Proof. The main idea is to compare the zeros of $\tilde{T}(q^\ell z)p_0(q^{-1}z)$ with the zeros of $\tilde{T}(q^{-\ell}z)p_0(qz)$:

$$\begin{aligned} \tilde{T}(q^\ell z)p_0(q^{-1}z) &\propto (z - q^{-1})(z - q^{2s_1}) \prod_{\alpha=0}^{s_1-1} (z - q^{2\alpha+1}) \\ &= (z - q^{2s_1}) \prod_{\alpha=0}^{s_1} (z - q^{2\alpha-1}) \\ \tilde{T}(q^{-\ell}z)p_0(qz) &\propto (z - q^{-2})(z - q^{2s_1-1}) \prod_{\alpha=0}^{s_1-1} (z - q^{2\alpha-1}) \\ &= (z - q^{-2}) \prod_{\alpha=0}^{s_1} (z - q^{2\alpha-1}) \end{aligned}$$

So we can form $\ell n - s_1$ polynomials that vanish when we evaluate in $\tilde{\mathbf{y}}$, i.e. $\langle \tilde{R}_a, \chi_b \rangle$ vanish when χ_b corresponds to $\{q^{-1}, \dots, q^{2s_1-1}\}$. So, we have $s_1 + 1$ rows vanishing in all columns except s_1 of them. The remaining rectangle $s_1 \times (s_1 + 1)$ is of rank at most s_1 , and therefore the $s_1 + 1$ rows are of rank at most s_1 and the determinant vanishes. □

A.9. The XXY wheel condition. Once again, we could use the symmetry to prove the XXY wheel condition. But we shall prove it using a different method.

For simplicity, set $x_1 = 1$, $y_1 = q^{1+2s_1}$ and $y_2 = q^{2+2s_1+2s_2}$. We start with a proposition similar to A.6:

Proposition A.8. *Consider the union of the two sets $\{q^{1+2s_1+2\alpha}\}_\alpha$ and $\{q^{2+2s_1+2s_2+2\alpha}\}_\alpha$. Then $\tilde{\mathbf{y}} = \{q^{-1}, q, q^3, \dots, q^{2\ell-1}\}$ is a subset of the union.*

Proof. We know that $s_1 \leq \ell - 1$, than all terms in $\tilde{\mathbf{y}}$ after q^{1+2s_1} appear there. Notice that in the worst case, $s_1 = 0$, $\{q^{1+2\alpha}\}_\alpha = \{q, q^3, \dots, q^{2\ell-1}\}$.

Let us see what happen in the two extreme for s_2 . If $s_2 = 0$, we easily get $\{q^{2+2s_1}, \dots, q^{2\ell+2s_1}\} = \{q^{2+2s_1}, \dots, q^{2s_1-1}\}$, notice that $2+2s_1 \leq 2\ell$, therefore this passes at q^{-1} . In the other extreme $s_1 + s_2 = \ell - 1$, than we obtain $\{q^{2\ell}, q, \dots, q^{2\ell-3}\}$. We are done. \square

Let $p_0(z) = \prod_{\alpha=0}^{\ell-1} (z - q^{2\alpha})$, it has degree ℓ , and then we can consider the polynomials below spanning an $\ell(n-1)$ dimensional subspace of $\tilde{\mathcal{S}}$:

$$\tilde{R}_a(z) = \frac{\tilde{T}(q^\ell z)p_a(q^{-1}z)p_0(q^{-1}z) - \tilde{T}(q^{-\ell}z)p_a(qz)p_0(qz)}{(q - q^{-1})z}$$

And, of course, we have ℓ dimensions left from $\tilde{\mathcal{S}}$.

Lemma A.9. *The XYY wheel condition holds.*

Proof. We repeat the exact same procedure as in the XXY wheel condition. We compare the zeros of $\tilde{T}(q^\ell z)p_0(q^{-1}z)$ with the zeros of $\tilde{T}(q^{-\ell}z)p_0(qz)$:

$$\begin{aligned} \tilde{T}(q^\ell z)p_0(q^{-1}z) &\propto (z - q^{-1}) \prod_{\alpha=0}^{\ell-1} (z - q^{2\alpha+1}) = \prod_{\alpha=0}^{\ell} (z - q^{2\alpha-1}) \\ \tilde{T}(q^{-\ell}z)p_0(qz) &\propto (z - q^{-2}) \prod_{\alpha=0}^{\ell-1} (z - q^{2\alpha-1}) = \prod_{\alpha=0}^{\ell} (z - q^{2\alpha-1}) \end{aligned}$$

So we can form $\ell(n-1)$ polynomials that vanish when we evaluate in $\tilde{\mathbf{y}}$, i.e. $\langle \tilde{R}_a, \chi_b \rangle$ vanish when χ_b corresponds to $\{q^{-1}, \dots, q^{2\ell-1}\}$. So, we have $\ell+1$ rows vanishing in all columns except ℓ of them. The remaining rectangle $\ell \times (\ell+1)$ is of rank at most ℓ , and therefore the $\ell+1$ rows are of rank at most ℓ and the determinant vanishes. \square

This is perhaps a good start to generalizations. And notice that the three proofs are similar, mainly the XXY and XYY wheel conditions.

APPENDIX B. PROOF OF UNIQUENESS

In this section, we will prove Lemma 5.10.

Recall, that in our case, when $q^{2l+1} = 1$, $p = q^2$ and $t = q$, the wheel condition simplifies to:

Definition B.1 (Simplified wheel condition). A polynomial $\mathcal{P}(z)$ is said to obey the simplified wheel condition if:

$$\mathcal{P}(z) = 0 \quad \text{if } z_k = q^{1+2s_2}z_j = q^{2+2s_1+2s_2}z_i$$

for all $s_1, s_2 \in \mathbb{N}$ such that $s_1 + s_2 \leq l-1$ and for any choice $0 < i < j < k \leq 2n$.

For the proof of the lemma, we need also a new definition, where we ignore relations like $z_j = q^{1+2s}z_i$ (as well as $z_k = q^{1+2s}z_j$ and $z_i = q^{1+2s}z_k$), for $s < m$:

Definition B.2 (r -wheel condition). This is exactly like the simplified wheel condition, except that we only require $z_k = q^{1+2r+2s_2}z_j = q^{2+4r+2s_1+2s_2}z_i$ with $s_1, s_2 \in \mathbb{N}$ such that $s_1 + s_2 \leq \ell - 1 - 3r$.

The more delicate point of the proof can be summarized by:

Lemma B.3. *Let \mathcal{R}_n be a polynomial in V_n , and let x_i and y_j be two variables, such that $\mathcal{R}_n|_{y_j=qx_i} = 0$, then \mathcal{R}_n vanishes identically.*

Proof. We assume $n > 1$, otherwise \mathcal{R}_n is a constant and the result is trivial. It is obvious that $\mathcal{R}_n \propto (y_j - qx_i)$, then by symmetry we can write

$$\mathcal{R}_n = \prod_{i,j} (y_j - qx_i)(x_i - qy_j) \mathcal{O}_n$$

where \mathcal{O}_n has degree $\delta = (\ell-2)n(n-1) - 2n$ and partial degree $\delta_i = (\ell-2)(n-1) - 2$. If $\ell \leq 2$, then \mathcal{O}_n vanishes identically, which forces \mathcal{R}_n also to vanish and we are done.

Otherwise, look at the consequence of the wheel condition in \mathcal{O}_n :

$$\mathcal{O}_n(\mathbf{x}, \mathbf{y})|_{x_j=qx_i} = \prod_{k \neq i,j} \prod_{s=1}^{\ell} (x_k - q^{2s} x_i) \prod_{k \neq i,j} \prod_{s=2}^{\ell-1} (y_k - q^{2s} x_i) \mathcal{O}'_n(\mathbf{x}, \mathbf{y})$$

when we count the degree of \mathcal{O}'_n in x_i and x_j , we obtain $-2n$, which is impossible, then $\mathcal{O}'_n = 0$. Therefore:

$$\mathcal{R}_n(\mathbf{z}) = \prod_{i \neq j}^{2n} (z_i - qz_j) \mathcal{R}_n^{(1)}(\mathbf{z})$$

where $\mathcal{R}_n^{(1)}$ obeys the 1-wheel condition. Its total degree is $\delta^{(1)} = (\ell-4)n(n-1) - 2n$ and the partial degree is $\delta_i^{(1)} = (\ell-4)(n-1) - 2$.

The 1-wheel condition⁷ implies that:

$$\mathcal{R}_n^{(1)}|_{z_j=q^3 z_i} = \prod_{k \neq i,j} \prod_{s=0}^{\ell-4} (z_k - q^{6+2s} z_i) \mathcal{R}_n'^{(1)}$$

but, when we count the degree of z_i and z_j in $\mathcal{R}_n'^{(1)}$ we obtain $-2(n-1) - 4$, and therefore it vanishes.

Therefore we can write

$$\mathcal{R}_n^{(1)}(\mathbf{z}) = \prod_{i \neq j}^{2n} (z_i - q^3 z_j) \mathcal{R}_n^{(2)}(\mathbf{z})$$

and we repeat this process several times.

In step r we have the polynomial $\mathcal{R}_n^{(r)}$, which obeys the r -wheel condition, it has total degree $\delta^{(r)} = (\ell-4r)n(n-1) - 2rn$ and partial degree $\delta_i^{(r)} = (\ell-4r)(n-1) - 2r$. Then the r -wheel condition implies:

$$\mathcal{R}_n^{(r)}|_{z_j=q^{1+2r} z_i} = \prod_{k \neq i,j} \prod_{s=0}^{\ell-1-3r} (z_k - q^{2+4r+2s} z_i) \mathcal{R}_n'^{(r)}$$

then the degree of z_i and z_j in $\mathcal{R}_n'^{(r)}$ is equal to $-2r(n-1) - 4r$, which is negative and therefore, we can iterate:

$$\mathcal{R}_n^{(r)}(\mathbf{z}) = \prod_{i \neq j}^{2n} (z_i - q^{1+2r} z_j) \mathcal{R}_n^{(r+1)}(\mathbf{z})$$

Now, the polynomial $\mathcal{R}_n^{(r+1)}(\mathbf{z})$ has total degree $\delta^{(r+1)} = (\ell-4(r+1))n(n-1) - 2(r+1)n$, partial degree $\delta_i^{(r+1)} = (\ell-4r)(n-1) - 2r$ and it obeys the $(r+1)$ -wheel condition.

At a certain r , $\delta^{(r)}$ will become negative, and therefore $\mathcal{R}_n^{(r)} = 0$, which propagates all way up to $\mathcal{R}_n = 0$. Notice that, all steps are well defined, because the

⁷Notice that for any choice of z_i and z_j , we can apply the available symmetries, such that they become neighbors

r -wheel condition, which requires $r \leq (\ell-1)/3$, still makes sense even when the degree $\delta^{(r)}$ becomes negative. Proving the result. \square

We can now prove the main result of this section: Lemma 5.10, which states that the space V_n is at most one dimensional.

Proof. We use induction on n in order to prove it. Call \mathcal{P}_n an arbitrary element of V_n . If $n = 1$, then \mathcal{P}_1 is a constant.

Suppose now that the uniqueness holds for $n - 1$. By the wheel condition (and by some symmetry considerations):

$$\mathcal{P}_n|_{y_j=qx_i} = \prod_{k \neq i} \prod_{s=1}^{\ell} (x_k - q^{2s}x_i) \prod_{k \neq j} \prod_{s=1}^{\ell} (y_k - q^{2s}x_i) \hat{\mathcal{P}}_n$$

We can make a list of the properties of $\hat{\mathcal{P}}_n$:

- The total degree is $\delta = \ell(n-1)(n-2)$.
- It does not depend on the variables x_i and y_j .
- The partial degree in the rest of the variables is $\delta_i = \ell(n-2)$.
- Exclude x_i and y_j from \mathbf{x} and \mathbf{y} and call it $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, respectively, then it obeys the simplified wheel condition on $\hat{\mathbf{z}} = \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$.
- It is symmetric in $\hat{\mathbf{x}}$, in $\hat{\mathbf{y}}$ and in the exchange $\hat{\mathbf{x}} \leftrightarrow \hat{\mathbf{y}}$.

therefore, by the recursion hypothesis, it is proportional to the unique $\mathcal{P}_{n-1}(\hat{\mathbf{z}})$. Eventually, $\hat{\mathcal{P}}_n$ can vanish, but in this case, Lemma B.3 implies that \mathcal{P}_n vanishes.

Pick now a second polynomial in V_n . By the above result, exists α such that

$$\mathcal{P}_n|_{y_j=qx_i} = \alpha \mathcal{Q}_n|_{y_j=qx_i}$$

or, in a more explicit form

$$(\mathcal{P}_n - \alpha \mathcal{Q}_n)|_{y_j=qx_i} = 0.$$

By Lemma B.3 the polynomial $\mathcal{P}_n - \alpha \mathcal{Q}_n$ vanishes and the two polynomials are linearly dependent. \square

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